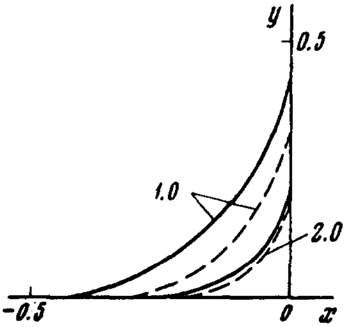

Fig. 1

symmetry and will consist of a number of equal sectors each of which can be mapped on a rectangle in the hodograph plane. Thus it is easy to obtain an approximate flow pattern and stagnation zones in the isolated regions. The outer stagnation zones surrounding the point at infinity are, of course, the most interesting ones, since the stagnation zones near the inner critical points of the flow are small, when λ is small.

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BIBLIOGRAPHY

1. Entov, V. M., On some two-dimensional problems of the theory of filtration with a limiting gradient. PMM Vol. 31, №5, 1967.
2. Entov, V. M., On one of filtration problems with a limit gradient allowing an exact solution. PMM Vol. 32, №3, 1968.
3. Lebedev, N. N., Special Functions and their Applications, 2-nd ed., M.-L., Fizmatgiz, 1963.
4. Sneddon, I., Fourier Transforms. Mc. Graw Hill, 1951.
5. Lavrent'ev, M. A. and Shabat, B. V., Methods of.the Theory of Functions of the Complex Variable, 2nd ed., M., Fizmatgiz, 1958.
6. Bateman, H. and Erdelyi, A. (ed.), Higher Transcendental Functions. Mc. Graw Hill.

Translated by L. K.

ANALYSIS OF SECONDARY STEADY FLOW BETWEEN ROTATING CYLINDERS

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Appearance of a secondary steady flow which is a Taylor vortex, caused by the loss of stability of the Couette flow between the rotating (in the same direction) cylinders is investigated using the Liapunov-Schmidt method. It is shown that the secondary solution can be obtained in the form of a series in powers of the parameter $\varepsilon = (N_{\rm Re} - N_{\rm Re*})^{1/2}$ where $N_{\rm Re}$ is the Reynolds' number and $N_{\rm Re*}$ denotes its critical value. First two terms of the series are analised for two separate cases and it is established that the Taylor vortex is defined uniquely with the accuracy of up to the displacement in the axial direction. Perturbation theory is used to show that at small ε the Taylor flow is stable with

respect to rotationally symmetric perturbations.

Example of analysis of the torque is given at the end of the paper.

1. Statement of the problem and the results. Let a viscous incompressible fluid of unit density and viscosity coefficient ν, fill the space between two concentric cylinders of radii R_1 and R_2 rotating with angular velocities Ω_1 and Ω_2 ($\Omega_1\Omega_2 > 0$). We shall seek the $2\pi/\alpha$-periodic flows parallel to the z-axis of the cylinders, under the assumption that no loss of fluid occurs on the transverse direction. Then the Navier-Stokes equations will have a solution $\mathbf{v}_0$ (Couette flow) whose components will be given, in cylindrical (r, θ, z)-coordinates, by

$$v_{0r} = v_{0z} = 0, \qquad v_{0\theta} = ar + b/r \tag{1.1}$$

$$a = \frac{R^2\Omega - 1}{R^2 - 1}, \quad b = -\frac{R^2(\Omega - 1)}{R^2 - 1}, \quad R = \frac{R_2}{R_1}, \quad \Omega = \frac{\Omega_2}{\Omega_1}$$

The Couette flow is defined by the parameters Ω and R and is independent of the Reynolds' number $N_{\mathrm{Re}} = \Omega_1 R_1^2/\nu$.

We shall seek a secondary flow $\mathbf{v}'$ in the form $\mathbf{v}' = \mathbf{v}_0 + \mathbf{v}$ and the corresponding pressure p' in the form $p' = p_0 + p/\mathrm{Re}$, where p_0 is the pressure corresponding to the flow (1.1). Quantities $\mathbf{v}$ and p are given by the following Eqs. (in dimensionless variables)

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \tag{1.2}$$

$$\Delta v_r - \frac{v_r}{r^2} - \frac{\partial p}{\partial r} = N_{\mathrm{Re}}\left[v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} - 2\frac{v_{0\theta}}{r}v_\theta\right] \tag{1.3}$$

$$\Delta v_\theta - \frac{v_\theta}{r^2} = N_{\mathrm{Re}}\left[v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} + \left(\frac{dv_{0\theta}}{dr} + \frac{v_{0\theta}}{r}\right)v_r\right] \tag{1.4}$$

$$\Delta v_z - \frac{\partial p}{\partial z} = N_{\mathrm{Re}}\left[v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right] \tag{1.5}$$

Solution $\mathbf{v}, p$ of the system (1.2)–(1.5) must be $2\pi/\alpha$-periodic in z and satisfy the following boundary conditions

$$\mathbf{v} = 0 \quad \text{when } r = 1, R \tag{1.6}$$

$$\int_1^R v_z(r, z)\, r\, dr = 0 \tag{1.7}$$

We shall assume that $a < 0$ and this will imply that the flow (1.1) is unstable at large N_{Re}.

Rigorous proof of the above statement, known already to Taylor (in 1924), is given in [1, 2 and 3]. Let $N_{\mathrm{Re}*}$ be the least eigenvalue of the corresponding linearized problem. It was shown in [2 and 3] that this eigenvalue is double for all α, with the exception of a certain enumerable set (it is single in the vector subspace in which v_r and v_θ are even and v_z is odd) and, that it represents a bifurcation point of the nonlinear problem (1.2)–(1.7)(see also [4 and 5]); when N_{Re} is almost equal to $N_{\mathrm{Re}*}$, then the above problem has nonzero solutions vanishing as $N_{\mathrm{Re}} \to N_{\mathrm{Re}*}$.

Subsequent application of the Liapunov-Schmidt method makes it possible to establish the number of small-value nonzero solutions and to investigate the spectral distribution of the nonlinear problem.

Using the results of [6] we show, that the nonzero solution is unique up to within the displacement along the z-axis (two solutions are obtained in the subspace indicated above) and, that it is an analytic function of the parameter $\varepsilon = (N_{\mathrm{Re}} - N_{\mathrm{Re}*})^{1/2}$. It exists only for $N_{\mathrm{Re}} > N_{\mathrm{Re}*}$ and is stable under the rotationally symmetric perturbations while the Couette flow (1.1) becomes unstable.

No general proof of the above facts exists (it can be given for the case of a narrow gap between the cylinders; see also analogous results obtained for the related free convection problem [6–8]). The main difficulty lies in the fact that the results of [6] can be utilized if and only if it is proved that a certain constant g is positive (see below [2.28]). This has not been proved for the general case but, when R and α are given, then its correctness can be established by numerical methods while computing the secondary flow. An example of such a procedure is given in Section 3.

To dispel the possible wrong impressions, we shall remind the readers that the "principle of alteration of stability" for the case of the Couette flow and the possibility of restricting ourselves to the rotationally symmetric perturbations (there is no analog to the Squire's theorem) although confirmed by experiments, have not yet received a rigorous proof.

2. The Liapunov-Schmidt method. 1. Liapunov-Schmidt series. We shall seek the solution of the problem (1.2) – (1.7) in the form

$$\mathbf{v} = \sum_{k=1}^{\infty} \varepsilon^k \mathbf{v}_k, \quad p = \sum_{k=1}^{\infty} \varepsilon^k p_k, \quad \varepsilon = (N_{\mathrm{Re}} - N_{\mathrm{Re}*})^{1/2} \tag{2.1}$$

Vectors $\mathbf{v}_k$ should be solenoidal, periodic in z, vanish at $r = 1, R$, their transverse flux should be equal to zero, and they should satisfy the equations following from (1.3) – (1.5) and (2.1)

$$A\mathbf{v}_k - \nabla p_k = N_{\mathrm{Re}*} \sum_{m+n=k} L(\mathbf{v}_m \mathbf{v}_n) + \sum_{m+n=k-2} L(\mathbf{v}_m, \mathbf{v}_n) + K\mathbf{v}_{k-2} \equiv \mathbf{f}_k \tag{2.2}$$

where the differential operators A and L are given for any vectors $\mathbf{u}$ and $\mathbf{v}$ symmetric under rotation, by
$$A = A_0 - N_{\mathrm{Re}*} K$$

$$(A_0\mathbf{v})_r = \Delta v_r - \frac{v_r}{r^2}, \quad (A_0\mathbf{v})_\theta = \Delta v_\theta - \frac{v_\theta}{r^2}, \quad (A_0\mathbf{v})_z = \Delta v_z$$

$$(K\mathbf{v})_r = -2\frac{v_{0\theta}}{r} v_\theta, \quad (K\mathbf{v})_\theta = \left(\frac{dv_{0\theta}}{dr} + \frac{v_{0\theta}}{r}\right) v_r, \quad (K\mathbf{v})_z = 0 \tag{2.3}$$

$$\{L(\mathbf{u}\mathbf{v})\}_r = -\frac{u_\theta v_\theta}{r} + u_r \frac{\partial v_r}{\partial r} + u_z \frac{\partial v_r}{\partial z}$$

$$\{L(\mathbf{u}, \mathbf{v})\}_\theta = \frac{u_\theta v_r}{r} + u_r \frac{\partial v_\theta}{\partial r} + u_z \frac{\partial v_\theta}{\partial z}, \quad \{L(\mathbf{u}, \mathbf{v})\}_z = u_r \frac{\partial v_z}{\partial r} + u_z \frac{\partial v_z}{\partial z}$$

We shall seek such solutions of (1.2) – (1.7), for which v_r and v_θ are even functions and v_z is an odd function of z, and therefore we shall subject the vectors $\mathbf{v}_k$ to the same conditions. In particular for $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ we have

$$A\mathbf{v}_1 - \nabla p_1 = 0, \quad A\mathbf{v}_2 - \nabla p_2 = N_{\mathrm{Re}*} L(\mathbf{v}_1, \mathbf{v}_1) \tag{2.4}$$

$$A\mathbf{v}_3 - \nabla p_3 = N_{\mathrm{Re}*} [L(\mathbf{v}_1, \mathbf{v}_2) + L(\mathbf{v}_2, \mathbf{v}_1)] + K\mathbf{v}_1 \tag{2.5}$$

The right-hand side of (2.2) contains only the coefficients of the expansion (2.1) whose indices are less than k. Therefore $\mathbf{v}_k$ and p_k can be determined from the consecutive solutions of the linear problems. For $\mathbf{v}_1$ and p_1 we have

$$\mathbf{v}_1 = \beta_1 \boldsymbol{\varphi}, \qquad p_1 = \beta_1 p_{10} \tag{2.6}$$

where β_1 is an independent constant and $\boldsymbol{\varphi}$, p_{10} represent the characteristic solution of the following boundary value problem:

$$A\boldsymbol{\varphi} - \nabla p_{10} = 0, \qquad \operatorname{div} \boldsymbol{\varphi} = 0 \tag{2.7}$$

$$\boldsymbol{\varphi}|_{r=1,\,R} = 0, \int_1^R \varphi_z r dr = 0 \tag{2.8}$$

This solution is $2\pi/\alpha$-periodic and such, that φ_r and φ_θ are even functions, while φ_z is an odd function of z. We shall include, for definiteness, the normalizing condition

$$\int_{-\pi/2\alpha}^{\pi/2\alpha} \int_1^R \varphi_r \, r dr \, dz = \frac{2}{\alpha} \tag{2.9}$$

Let $\boldsymbol{\psi}$ and q be a characteristic solution of the conjugate problem

$$A^*\boldsymbol{\psi} - \nabla q = 0, \qquad \operatorname{div} \boldsymbol{\psi} = 0 \tag{2.10}$$

$$\boldsymbol{\psi}|_{r=1,\,R} = 0, \qquad \int_1^R \psi_z \, r dr = 0 \tag{2.11}$$

$$A^* = A_0 - N_{\mathrm{Re}^*} K^*, \qquad \{K^*\mathbf{v}\}_r = \left(\frac{dv_{0\theta}}{dr} + \frac{v_{0\theta}}{r}\right) v_\theta$$

$$\{K^*\mathbf{v}\}_\theta = -2\frac{v_{0\theta}}{r} v_r, \qquad \{K^*\mathbf{v}\}_z = 0$$

satisfying the previous conditions of periodicity, parity and the normalizing conditions. Then the condition that the boundary value problem (2.2) has a solution, assumes the form [2]

$$\int_0^{2\pi/\alpha} \int_1^R \mathbf{f}_k \boldsymbol{\psi} \, r dr dz = 0 \tag{2.12}$$

2. Linearized problem. We shall seek the solution $\boldsymbol{\varphi}$ of the problem (2.7), (2.8) in the form

$$\varphi_r(r, z) = \varphi_{1r}(r) \cos \alpha z, \qquad \varphi_\theta(r, z) = \varphi_{1\theta}(r) \cos \alpha z$$

$$\varphi_z(r, z) = \varphi_{1z}(r) \sin \alpha z, \qquad p_{10}(r, z) = q_1(r) \cos \alpha z \tag{2.13}$$

Functions φ_{1r} and $\varphi_{1\theta}$ will be solutions of the following system of ordinary differential equations

$$(L - \alpha^2)^2 \varphi_{1r} = 2\alpha^2 \lambda \omega(r) \varphi_{1\theta}, \quad (L - \alpha^2) \varphi_{1\theta} = 2a\lambda \varphi_{1r} \tag{2.14}$$

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}, \qquad \omega(r) = a + \frac{b}{r^2} \tag{2.15}$$

and will satisfy the boundary and normalizing conditions

$$\varphi_{1r} = \frac{d\varphi_{1r}}{dr} = \varphi_{1\theta} = 0 \quad \text{when } r = 1, R; \qquad \int_1^R \varphi_{1r} \, r dr = 1 \tag{2.16}$$

Functions φ_{1z} and q_1 are given by

$$\varphi_{1z} = -\frac{1}{\alpha r}\frac{d}{dr}(r\varphi_{1r}), \qquad q_1 = -\frac{1}{\alpha}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \alpha^2\right)\varphi_{1z} \tag{2.17}$$

It was shown in [3] that for any α the problem (2.14) – (2.16) has a sequence of positive and simple eigenvalues $0 < \lambda_1(\alpha) < \lambda_2(\alpha) < \ldots$. Moreover, for all α except some enumerable set, $N_{\mathrm{Re}^*} = \lambda_1(\alpha)$ is a simple eigenvalue of the problem (2.7), (2.8).

We shall further assume that α is not a member of the above set. Then $\lambda_1(\alpha) < \lambda_k(m\alpha)$ for all natural k and m except $k = m = 1$.

We similarly obtain the solution of the conjugate system (2.10), (2.11) in the form

$$\psi_r(r, z) = \psi_{1r}(r) \cos\alpha z, \quad \psi_\theta(r, z) = \psi_{1\theta}(r) \cos\alpha z \qquad (2.18)$$

$$\psi_z(r, z) = \psi_{1r}(r) \sin\alpha z, \quad q(r, z) = q_0(r) \cos\alpha z$$

The following boundary value problem yields $\psi_{1z}(r)$ and $\psi_{1\theta}(r)$

$$(L - \alpha^2)^2 \psi_{1r} = -2\alpha^2 a N_{\mathrm{Re}^*}\psi_{1\theta}, \quad (L - \alpha^2)\psi_{1\theta} = -2N_{\mathrm{Re}^*}\omega\, \psi_{1r} \qquad (2.19)$$

$$\psi_{1r} = \frac{d\psi_{1r}}{dr} = \psi_{1\theta} = 0 \text{ when } r = 1, R; \qquad \int_1^R \psi_{1\theta}\, r dr = 1 \qquad (2.20)$$

Functions ψ_{1z} and q_0 are given by

$$\psi_{1z} = -\frac{1}{\alpha r}\frac{d}{dr}(r\psi_{1r}), \qquad q_0 = -\frac{1}{\alpha}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \alpha^2\right)\psi_{1z} \qquad (2.21)$$

3. Determination of v_2. From (2.4) and (2.6) we have

$$A\mathbf{v}_2 - \nabla p_2 = N_{\mathrm{Re}^*}\beta_1^2 L(\psi, \varphi) \qquad (2.22)$$

and we seek its solution in the form

$$\mathbf{v}_2 = \beta_2\varphi + N_{\mathrm{Re}^*}\beta_1^2 \mathbf{w}, \quad p_2 = \beta_2 p_1 + N_{\mathrm{Re}^*}\beta_1^2 q_2 \qquad (2.23)$$

where β_2 is an unknown constant while $\mathbf{w}$ and q_2 represent the solution of the following boundary problem:

$$\Delta w_r - \frac{w_r}{r^2} + 2N_{\mathrm{Re}^*}\,\omega w_\theta = \frac{\partial q_2}{\partial r} + \frac{1}{2}F_1(r) + \frac{1}{2}F_2(r)\cos 2\alpha z$$

$$\Delta w_\theta - \frac{w_\theta}{r^2} - 2aN_{\mathrm{Re}^*} w_r = \frac{1}{2}F_3(r) + \frac{1}{2}F_4(r)\cos 2\alpha z \qquad (2.24)$$

$$\Delta w_z = \frac{\partial q_2}{\partial z} + \frac{1}{2}F_5(r)\sin 2\alpha z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rw_r) + \frac{\partial w_z}{\partial z} = 0, \quad w_r = w_\theta = w_z = 0 \text{ when } r = 1, R$$

Here we utilize the following notation:

$$F_1(r) = -\frac{\varphi_{1\theta}^2}{r} + \varphi_{1r}\frac{d\varphi_{1r}}{dr} - \alpha\varphi_{1z}\varphi_{1r}, \quad F_2(r) = -\frac{\varphi_{1\theta}^2}{r} + \varphi_{1r}\frac{d\varphi_{1r}}{dr} + \alpha\varphi_{1z}\varphi_{1r}$$

$$F_3(r) = \frac{\varphi_{1\theta}\varphi_{1r}}{r} + \varphi_{1r}\frac{d\varphi_{1\theta}}{dr} - \alpha\varphi_{1z}\varphi_{1\theta}, \quad F_4(r) = \frac{\varphi_{1\theta}\varphi_{1r}}{r} + \varphi_{1r}\frac{d\varphi_{1\theta}}{dr} + \alpha\varphi_{1z}\varphi_{1r}$$

$$F_5(r) = \varphi_{1r}\frac{d\varphi_{1z}}{dr} + \alpha\varphi_{1z}^2$$

Solution of (2.24) has the form

$$w_r(r, z) = w_{1r}(r)\cos 2\alpha z, \quad w_\theta(r, z) = w_{0\theta}(r) + w_{1\theta}(r)\cos 2\alpha z \qquad (2.25)$$

$$w_z(r, z) = w_{1z}(r)\sin 2\alpha z, \quad q_2(r, z) = q_{20}(r) + q_{21}(r)\cos 2\alpha z$$

and the solution of the boundary value problem

$$(L - 4\alpha^2)^2 w_{1r} = 8N_{\mathrm{Re}^*}\alpha^2\omega w_{1\theta} - 2\alpha^2 F_2 - \alpha\frac{dF_5}{dr} \qquad (2.26)$$

$$(L - 4\alpha^2) w_{1\theta} = 2aN_{\mathrm{Re}^*} w_{1r} + \frac{F_4}{2}, \quad w_{1r} = \frac{dw_{1r}}{dr} = w_{1\theta} = 0 \text{ when } r = 1, R$$

yields the functions w_{1r} and $w_{1\theta}$ while for $w_{0\theta}$, w_{1z}, q_{20} and q_{21} we obtain

$$w_{0\theta} = \frac{r}{2}\int_1^r \frac{\varphi_{1\theta}(\rho)\,\varphi_{1r}(\rho)}{\rho}\,d\rho - \frac{(r^2-1)\,R^2}{2r\,(R^2-1)}\int_1^R \frac{\varphi_{1\theta}(\rho)\,\varphi_{1r}(\rho)}{\rho}\,d\rho$$

$$w_{1z} = -\frac{1}{2\alpha r}\frac{d}{dr}(rw_{1r}), \qquad q_{20} = \int_1^r \left(2N_{\mathrm{Re}*}\,\omega w_{0\theta} - \frac{F_1}{2}\right) d\rho + \text{const}$$

$$q_{21} = -\frac{1}{2\alpha}\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 4\alpha^2\right) w_{1z} - \frac{F_5}{2}\right] \tag{2.27}$$

The system (2.26) has a unique solution.

Indeed, as it was shown previously, $N_{\mathrm{Re}*} = \lambda_1(\alpha)$, and it cannot coincide with any of its eigenvalues $\lambda_1(2\alpha)$, $\lambda_2(2\alpha)$... . Condition that (2.12) has a solution for $k = 3$ and $k = 4$, yields

$$\beta_1^2 = \frac{J_1}{J_2} \equiv g, \qquad \beta_2 = 0 \tag{2.28}$$

$$J_1 = \int_0^{2\pi/\alpha}\int_1^R \operatorname{rot}\varphi \operatorname{rot}\psi \; r\,dr\,dz \tag{2.29}$$

$$J_2 = N_{\mathrm{Re}*}^3 \int_0^{2\pi/\alpha}\int_1^R [L(\varphi, \mathbf{w}) + L(\mathbf{w}, \varphi)]\,\psi\, r\,dr\,dz \tag{2.30}$$

We note that if $\mathbf{v}_k$ is sought in the form $\mathbf{v}_k = \beta_k\varphi + \mathbf{w}_k$, where β_k is an unknown constant and $\mathbf{w}_k$ is a particular solution, orthogonal in ψ, of the corresponding inhomogeneous problem (2.2), then the solvability condition implies that all β_k with k even, vanish.

4. Spectral perturbation and stability. Small perturbation equation for the Couette flow has the form

$$\sigma\mathbf{u} - \frac{1}{N_{\mathrm{Re}}}\Delta\mathbf{u} + K\mathbf{u} = -\nabla p \tag{2.31}$$

When $N_{\mathrm{Re}} = N_{\mathrm{Re}*}$, then $\sigma = 0$ is one of the eigenvalues of the problem (2.31).

For the supercritical Reynolds' numbers $(N_{\mathrm{Re}} = N_{\mathrm{Re}*} + \varepsilon^2)$ the first eigenvalue σ can, in accordance with the perturbation theory, be expanded into the series

$$\sigma = \sigma_1\varepsilon^2 + \sigma_2\varepsilon^4 + \ldots, \quad \sigma_1 = \frac{J_1}{J_3}, \quad J_3 = N_{\mathrm{Re}*}\int_0^{2\pi/\alpha}\int_1^R \varphi\psi\, r\,dr\,dz \tag{2.32}$$

where J_1 is given by (2.29).

For the Taylor flow we have the following small perturbation equation:

$$\sigma'\mathbf{u} - \frac{1}{N_{\mathrm{Re}}}\Delta\mathbf{u} + K\mathbf{u} + \varepsilon\beta_1[L(\varphi, \mathbf{u}) + L(\mathbf{u}, \varphi)] +$$

$$+ \varepsilon\beta_1^2 N_{\mathrm{Re}*}[L(\mathbf{u}, \mathbf{w}) + L(\mathbf{w}, \mathbf{u})] + \ldots = -\nabla q \tag{2.33}$$

When the Reynolds' numbers are supercritical, the first eigenvalue σ' can be expanded into

$$\sigma' = \sigma_{12}\varepsilon^2 + \sigma_{13}\varepsilon^3 + \ldots, \qquad \sigma_{12} = -2\sigma_1$$

and, applying the result of [6], we can formulate the following theorem.

Theorem. Let the quantities g and σ_1 defined by (2.28) and (2.32) be positive. Then, as the Reynolds' number passes through its critical value $N_{\mathrm{Re}*} = \lambda_1(\alpha)$, the Couette flow becomes unstable. However, a new, stable steady flow appears (the Taylor vortex), which can be represented by the Liapunov-Schmidt series (2.1) and which is uniquely defined to within the displacement along the z-axis) by its wave number α.

Conditions of the theorem (positiveness of g and σ_1) have so far been verified only for the case of a narrow gap between the cylinders and for the allied problems on convection [5 – 7]. In the present paper the validity of these assumptions is established by numerical methods.

3. Computation of the Taylor vortex. Let us consider the following boundary value problem

$$(L-\alpha^2)u=f, \quad u=0 \quad \text{when } r=1, R \tag{3.1}$$

Its Green's function is given by

$$G_{1,\alpha}(r,\rho)=\begin{cases} D^{-1}\psi_1(r)\psi_2(\rho) & \text{for} \quad r\leqslant\rho \\ D^{-1}\psi_1(\rho)\psi_2(r) & \text{for} \quad r\geqslant\rho \end{cases} \tag{3.2}$$

where

$$D=-I_1(\alpha)K_1(\alpha R)+I_1(\alpha R)K_1(\alpha)$$

$$\psi_1(r)=I_1(\alpha r)K_1(\alpha)-I_1(\alpha)K_1(\alpha r)$$

$$\psi_2(r)=I_1(\alpha R)K_1(\alpha r)-K_1(\alpha R)I_1(\alpha r)$$

The boundary value problem

$$(L-\alpha^2)^2u=f, \quad u=\frac{du}{dr}=0 \quad \text{when } =1, R$$

has the corresponding Green's function $G_{2,\alpha}(r,\rho)$. Since it is symmetrical, the expression for $r\geqslant\rho$

$$G_{2,\alpha}(r,\rho)=\{K_1(\alpha r)-D_1^{-1}[\Lambda_3(R)u_1(r)-\Lambda_1(R)u_2(r)]\}u_1(\rho)+$$

$$+\{-I_1(\alpha r)+D_1^{-1}[\Lambda_1(R)u_1(r)-\Lambda_2(R)u_2(r)]\}u_2(\rho) \tag{3.3}$$

where

$$u_1=\Lambda_1(r)I_1(\alpha r)-\Lambda_2(r)K_1(\alpha r), \quad u_2=\Lambda_3(r)I_1(\alpha r)-\Lambda_1(r)K_1(\alpha r)$$

$$\Lambda_1(r)=\int_1^r I_1(\alpha r)K_1(\alpha r)\,rdr, \quad \Lambda_2(r)=\int_1^r I_1^2(\alpha r)\,rdr$$

$$\Lambda_3(r)=\int_1^r K_1^2(\alpha r)\,rdr, \quad D_1=\Lambda_1^2(R)-\Lambda_2(R)\Lambda_3(R)$$

is sufficient to define it.

Integrals $\Lambda_1(r)$, $\Lambda_2(r)$ and $\Lambda_3(r)$ can be expressed in the terms of Bessel functions (see e.g. [9], pp. 96-99).

Relations (2.14) – (2.16) yield

$$\varphi_{1r}=2N_{\mathrm{Re}*}\alpha^2\int_1^R G_{2,\alpha}(r,\rho)\,\omega(\rho)\,\varphi_{10}(\rho)\,\rho d\rho$$

$$\varphi_{10}=2aN_{\mathrm{Re}*}\int_1^R G_{1,\alpha}(r,\rho)\,\varphi_{1r}(\rho)\,\rho d\rho \tag{3.4}$$

from which we obtain

$$\varphi_{1r}=\lambda\int_1^R G_{3,\alpha}(r,\rho)\,\varphi_{1r}(\rho)\,\rho d\rho \tag{3.5}$$

$$\lambda=4N_{\mathrm{Re}*}^2\alpha^2, \quad G_{3,\alpha}(r,\rho)=a\int_1^R G_{2,\alpha}(r,s)\,G_{1,\alpha}(s,\rho)\,\omega(s)\,sds \tag{3.6}$$

The eigenvalue and the corresponding characteristic solution of the integral equation (3.5) can be found using the method of consecutive approximations according to the scheme

$$\lambda_{(n-1)} = \left(\int_1^R \int_1^R G_{3,\alpha}(r,\rho)\,\varphi_{1r(n-1)}(\rho)\,r\rho d\rho dr\right)^{-1} \qquad (3.7)$$

$$\varphi_{1r(n)} = \lambda_{(n-1)} \int_1^R G_{3,\alpha}(r,\rho)\,\varphi_{1r(n-1)}(\rho)\,\rho d\rho \qquad (3.8)$$

Since the kernel $G_{3,\alpha}(r,\rho)$ is oscillatory [3] and the theorem on the positiveness of its eigenvalue holds (see e.g. [10], ch. 2, Section 3), the sequences (3.7) and (3.8) converge, respectively, to the smallest eigenvalue and to the eigenfunction of Eq. (3.5).

The value of $\alpha = \alpha_*$ chosen for the computations was such, that $N_{Re^*} = \lambda_1(\alpha_*) = \min_\alpha \lambda_1(\alpha)$, since this case is the most interesting from the physical point of view. Computations can, however, be performed for any value of α.

Values of the kernels $G_{1,\alpha}, G_{2,\alpha}$ and $G_{3,\alpha}$ were given in the matrix form. All necessary integration was performed using the Simpson, trapezoidal or rectangular rule, and the interval of integration was divided into sixteen equal parts. The final results were accurate to at least two or three digits. The method converged very rapidly for the example chosen, and only six integrations were needed.

Having obtained φ_{1r}, we can find φ_{10} from (3.4), and φ_{1z} from (2.17), we obtain

$$\varphi_{1z} = -2N_{Re^*}\,\alpha \int_1^R \frac{1}{r}\left(G_{2,\alpha}(r,\rho) + r\frac{\partial G_{2,\alpha}(r,\rho)}{\partial r}\right)\omega(\rho)\,\varphi_{10}(\rho)\,\rho d\rho \quad (3.9)$$

The derivative $\partial G_{2,\alpha}(r,\rho)/\partial r$ and other necessary derivatives were calculated directly in order to eliminate any errors which might have arisen during the numerical differentiation. Tables 1 and 2 give the results of computation for two cases ($R = 2$ and $R = 1{,}5$).

The conjugate system (2.19), (2.20) can be replaced by the following integral equations

$$\psi_{1r} = -2N_{Re^*}\,a\alpha^2 \int_1^R G_{2,\alpha}(r,\rho)\,\psi_{10}(\rho)\,\rho d\rho \qquad (3.10)$$

$$\psi_{10} = -2N_{Re^*} \int_1^R G_{1,\alpha}(r,\rho)\,\omega(\rho)\,\psi_{1r}(\rho)\rho d\rho \qquad (3.11)$$

which yield the following expression :

$$\psi_{10} = 4N_{Re^*}^2\alpha^2 \int_1^R G_{3,\alpha}(\rho,r)\,\psi_{10}(\rho)\,\rho d\rho \qquad (3.12)$$

From (2.21) and (3.10) we obtain

$$\psi_{1z} = 2N_{Re^*}\,\alpha a \int_1^R \frac{1}{r}\left[G_{2,\alpha}(r,\rho) + r\frac{\partial G_{2,\alpha}(r,\rho)}{\partial r}\right]\psi_{10}(\rho)\,\rho d\rho \qquad (3.13)$$

Table 1

Results of computation for the case $R=2$, $\alpha=3,163$
$N_{\mathrm{Re}^*}=67.87$, $\beta_1=0.0188$, $\sigma_1=0.349$

r	φ_{1r}	φ_{10}	φ_{1z}	ψ_{1r}	$\psi_{1\theta}$	ψ_{1z}	w_{1r}	w_{00}	$w_{1\theta}$	w_{1z}
1,0625	0.1078	0.5638	—1.0195	0.0451	0.3241	—0.4285	0,0083	—0.0811	0.0052	—0.0187
1.125	0.3538	1.0098	—1.5182	0.1501	0.6245	—0.6593	0.0199	—0.1516	0.0107	—0,0327
1.1875	0.6432	1.5912	—1.6281	0.2789	0.8776	—0.7369	0.0340	—0.2027	0.0173	—0.0415
1.25	0.9126	2.0107	—1.4660	0.4050	1.0588	—0.7006	0.0487	—0.2263	0.0250	—0,0438
1.3125	1.1217	2.3332	—1.1344	0.5105	1.1685	—0.5831	0.0612	—0.2195	0.0333	—0.0397
1.375	1.2332	2.5402	—0.7180	0.5739	1.1974	—0.4139	0.0705	—0.1861	0,0417	—0.0305
1.4375	1.2918	2.6259	—0.2806	0.6200	1.1625	—0,2186	0.0750	—0.1341	0.0480	—0.0183
1.5	1.2473	2.5886	0.1302	0.6169	1.0752	—0.1891	0.0745	—0.0743	0.0542	—0.0053
1.5625	1.1141	2.4394	0.4805	0.5661	0.9507	0.1669	0.0697	—0.0192	0,0564	0.0068
1.625	0.9762	2.1991	0.7480	0.5070	0.8057	0.3237	0.0612	0.0237	0.0552	0.0170
1.6875	0.7767	1,8854	0.9186	0.4134	0.6528	0.4387	0.0594	0,0510	0.0503	0.0246
1.75	0.5625	1.5224	0.9335	0.3064	0.5022	0.5010	0.0383	0.0608	0.0425	0.0203
1.8125	0.3540	1.1345	0,9353	0.1975	0.3604	0.5000	0.0259	0.0560	0.0326	0.0308
1.875	0.1758	0,7427	0.7655	0.0908	0.2304	0.4250	0.0143	0.0411	0.0215	0.0274
1.9375	0,0488	0.3624	0.4607	0.0282	0.1112	0.2634	0.0050	0.0212	0.0104	0.0191

Table 2

Results of computation for the case $R=1.5$, $\alpha=6$
$N_{\mathrm{Re}^*}=152.4$, $\beta_1=0.0060$, $\sigma_1=0.696$

r	φ_{1r}	φ_{10}	φ_{1z}	ψ_{1r}	ψ_{10}	ψ_{1z}	w_{1r}	w_{00}	w_{10}	w_{1z}
1.0313	0.6641	1.9642	—2.1951	0.1262	0.2246	—1.2652	0.0003	—0.2369	—0.0144	—0.0086
1.0625	1.3020	3.7832	—3.3106	0.4327	0.7527	—2.0095	0.0227	—0.4273	—0.0155	—0.0402
1.0938	1.8665	5.3767	—3.6069	0.8267	1.3990	—2.3204	0.0466	—0.5408	0.0027	—0.0772
1.125	2.3144	6.6759	—3.2862	1.2335	2.0274	—2.2760	0.0795	—0.5616	0.0303	—0.1035
1.1563	2.6142	7.6299	—2.5704	1.5948	2.5434	—1.9522	0.1154	—0.4928	0.0714	—0.1107
1.1875	2.7537	8.2117	—1.6290	1.8677	2.8897	—1.4255	0.1481	—0.3546	0.1172	—0.0989
1.2188	2.7393	8.4189	—0.6091	2.0244	3.0400	—0.7735	0.1712	—0.1780	0.1585	—0.0707
1.25	2.5920	8.2716	0.3735	2.0520	2.9935	—0,0724	0.1815	0.0030	0.1871	—0.0321
1.2813	2.3417	7.8079	1.2324	1.9513	2.7695	0.6064	0.1587	0.1591	0.1970	0.0173
1.3125	2.0222	7.0780	1.9034	1.7358	2.4014	1.1959	0.1555	0.2702	0.1931	0.0497
1,3438	1.6662	6.1372	2.3416	1.4300	1.9328	1.6369	0.1282	0.3271	0.1736	0.0703
1.375	1.3005	5.0408	2.5219	1.0680	1.4139	1.8811	0.0741	0.3304	0.1426	0.0783
1.4063	0,9450	3.8395	2.4038	0.6916	0.8997	1.8778	0.0675	0.2874	0.1100	0.0817
1.4375	0.6100	2.5764	1,9720	0.3504	0.4489	1.5921	0.0383	0.2096	0.0726	0.07[illegible]0
1.4688	0.2069	1.2871	1.1890	0.0090	0.1253	0.0819	0.0138	0.1006	0.0345	0.0427

The conjugate problem can be solved using the method given above, and the results are given in the Tables 1 and 2.

From (2.26) we obtain the following expressions for w_{1r} and $w_{1\theta}$

$$w_{1r}=8N_{\mathrm{Re}^*}\alpha^2\int_1^R G_{2,2\alpha}(r,\rho)\,\omega(\rho)\,w_{1\theta}\rho d\rho-\int_1^R G_{2,2\alpha}(r,\rho)\left[2\alpha^2F_2+\alpha\frac{dF_5}{d\rho}\right]\rho d\rho \tag{3.14}$$

$$w_{1\theta}=2aN_{\mathrm{Re}^*}\int_1^R G_{1,2\alpha}(r,\rho)\,w_{1r}\,\rho d\rho+\int_1^R G_{1,2\alpha}(r,\rho)\,F_4(\rho)\,\rho d\rho \tag{3.15}$$

and

$$w_{1r}=\mu\int_1^R G_{3,2\alpha}(r,\rho)\,w_{1r}\rho d\rho+F(r),\qquad \mu=16\,N_{\mathrm{Re}^*}^2\alpha^2 \tag{3.16}$$

$$F(r)=8N_{\mathrm{Re}^*}^2\alpha^2\int_1^R G_{3,2\alpha}(r,\rho)\,F_4(\rho)\,\rho d\rho-\int_1^R G_{2,2\alpha}(r,\rho)\left[2\alpha^2F_2+\alpha\frac{dF_5}{d\rho}\right]\rho d\rho$$

Inhomogeneous integral equation (3.16) is solved by the method of consecutive approximations which converges, if $\mu < \mu_1$ (μ_1 is the smallest eigenvalue of the kernel $G_{3,2\alpha}$). This condition is fulfilled if α is sufficiently near α_* and for any $\alpha > \alpha_*$. Indeed, in this case we have

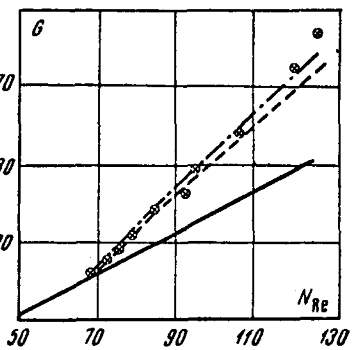

Fig. 1

$$\mu/\mu_1 = \lambda_1^2(\alpha)/\lambda_1^2(2\alpha) < 1$$

In particular, when $\alpha = \alpha_*$, $\mu/\mu_1 \approx 1/4$ and the method converges exceptionally well. Taking $w_{1r(0)} = F(r)$ for the null approximation, we obtain the desired result already on the twelfth iteration.

Function $w_{1\theta}$ is obtained from (3.15), while (2.27) and (3.14) together yield

$$w_{1z} = -\frac{1}{2\alpha}\int_1^R \frac{1}{r}\left[G_{2,2\alpha}(r,\rho) + r\frac{\partial G_{2,2\alpha}(r,\rho)}{\partial r}\right] \times$$
$$\times \left(8N_{\mathrm{Re}_*}\alpha^2\omega w_{1\theta} - 2\alpha^2 F_2 - \alpha\frac{dF_5}{d\rho}\right)\rho d\rho$$

Function $w_{0,\theta}$ together with the constants β_1 and σ_1 are obtained by numerical integration using Formulas (2.27) – (2.30) and (2.32).

Tables 1 and 2 give the results of the computations. Fig. 2 and 3 show the components of $\mathbf{v}_1$ and $\mathbf{v}_2$ as functions of the radius, and the stream function of the Taylor vortex.

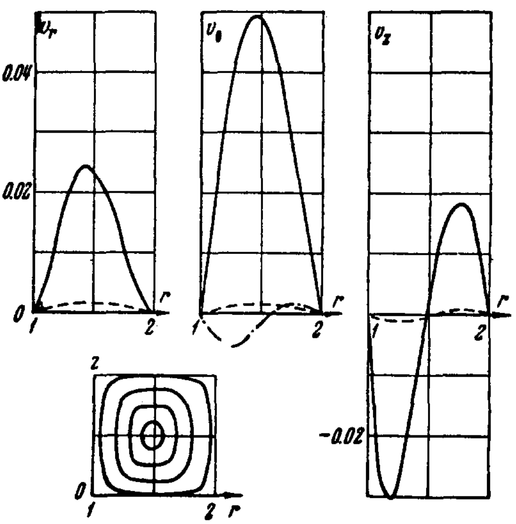

Fig. 2

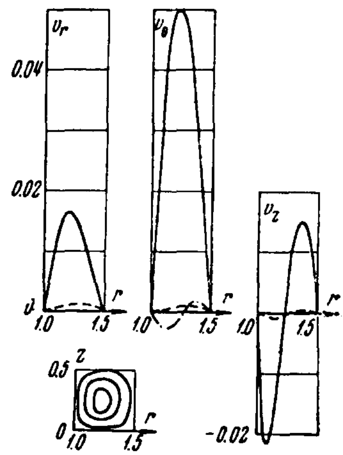

Fig. 3

Components of the vector $\beta_1\varphi_1$ where $\varphi_1 = \{\varphi_{1r}, \varphi_{1\theta}, \varphi_{1z}\}$ are shown with solid lines, components of the vector $\beta_1^2 N_{\mathrm{Re}*}\mathbf{w}_1$ where $\mathbf{w}_1 = \{w_{1r}, w_{1\theta}, w_{1z}\}$ are shown with dashed lines and the function $\beta_1^2 N_{\mathrm{Re}*}w_{0\theta}$ by the dot-dash line.

4. Evaluation of the torque. The torque due to the viscous forces acting on the inner cylinder is given by

$$G = \frac{2\pi R_1 h \rho \nu^2}{R_2 - R_1} N_{\mathrm{Re}} \int\limits_{1}^{2\pi/\alpha} \left(\frac{dv_0}{dr} - \frac{v_0}{r} \right) \Bigg|_{r=1} dz \qquad (4.1)$$

where h is the length of the cylinder. Inserting the expansion (2. 1) into (4. 1) we obtain the expression for G in the form of a series in powers of $\varepsilon^2 = N_{\mathrm{Re}} - N_{\mathrm{Re}*}$

$$G = G_0 + \varepsilon^2 G_2 + \varepsilon^4 G_4 + \ldots \qquad (4.2)$$

Thus the formation of the Taylor vortices leads to the appearance of an angular point on the graph of the function $G = G\,(N_{\mathrm{Re}})$.

Results of the computation of the viscous torque for $h = 5\,\mathrm{cm}$, $\nu = 0.1226\,\mathrm{cm}^2\,\mathrm{sec}^{-1}$, and $\rho = 0.8404\,\mathrm{cm}^{-3}$. For $R = 2$ and $\alpha = 3.163$ we have

$$G = 71.78 + 1.96\,\varepsilon^2 + \ldots \qquad (4.3)$$

while for $R = 1.5$ and $\alpha = 6$, we have

$$G = 435.4 + 8.18\,\varepsilon^2 + \ldots$$

Fig. 1 shows the comparison of the values of the torque calculated according to (4. 3), with the results of Donnelly and Simon [11] and the approximate values obtained by Davey in [12]. Solid line is used to plot the viscous torque for the Couette flow, broken line shows the results of Davey and the dot-dash line denotes the results of the present paper. Experimental values obtained by Donnelly and Simon are denoted by crosses. It should be noted that a good agreement with the experimental data is obtained over an unexpectedly wide interval of the Reynolds' numbers, and in any case, for $N_{\mathrm{Re}} \leqslant 120$. This confirms, that the terms neglected in (4. 3) were small. Presumably, even the appearance of the azimuthal waves resulting from the instability of the Taylor vortex, does not affect the torque to any significant degree.

BIBLIOGRAPHY

1. Krylov, A. L., Proof of the instability of a viscous, incompressible fluid flow. Dokl. Akad. Nauk SSSR, Vol. 153, №4, 1963.
2. Iudovich, V. I., Bifurcation of rotational fluid flows. Dokl. Akad. Nauk SSSR, Vol. 169, №2, 1966.
3. Iudovich, V. I., Secondary flows and fluid instability between rotating cylinders. PMM Vol. 30, №4, 1966.
4. Ivanilov, Iu. P. and Iakovlev, G. N., The bifurcation of fluid flow between rotating cylinders. PMM Vol. 30, №4, 1966.
5. Velte, W., Stabiliat und Verzweigung stationarer Losungen der Navier-Stokesschen Gleichungen beim Taylorproblem. Arch. Rational Mech. Anal., 22, №1, 1966.
6. Iudovich, V. I., Free convection and bifurcation. PMM Vol. 31, №1, 1967.
7. Iudovich, V. I., On the origin of convection. PMM Vol. 30, №6, 1966.
8. Iudovich, V. I., Stability of convection flows. PMM Vol. 31, №2, 1967.
9. Gray, A. and Mathews, G. B., Treatise on Bessel Functions and their Applications to Physics. 2nd ed., Peter Smith.
10. Krasnosel'skii, M. A., Positive Solutions of Operator Equations. M., Fizmatgiz, 1962.
11. Donnelly, R. J. and Simon, N. J., An empirical torque relation for supercritical flow between rotating cylinders. J. Fluid Mech., Vol. 7, pt. 3, pp. 401-418, 1960.
12. Davey, A., The growth of Taylor vortices in flow between rotating cylinders. J. Fluid Mech., Vol. 14, pt. 3, p. 336, 1962.

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